APPLICATION OF SLENDER BODY THEORY TO THE CALCULATION OF AERODYNAMIC PROPERTIES OF LOW ASPECT RATIO WINGS WITH NACELLES AT THEIR TIPS

(PRILOZHENIE TEORII TONKOGO TELA K RASCHETU AERODINAMICHESKIKH KHARAKTERISTIK KRYLA MALOGO UDLINENIIA S GONDOLAMI NA KONTSAKH)

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Lift and pitching moment coefficients of a plane wing of low aspect ratio with nacelles at its tips are determined by slender body theory and by application of the theory of functions of a complex variable [1]. Consideration is given to combinations shown on Fig.1, where 1 and 2 are leading and central parts of the wing, 3 and 4 are the leading and central parts of nacelles, 5 and 6 are leading and trailing edges of the wing.

The nacelles, which are bodies of revolution, have axes parallel to the axis of the wing and coplanar with it; the trailing edge of the wing is straight and is coplanar with the base of nacelles. The leading edge is intersected only at one point by a straight line parallel to the axis

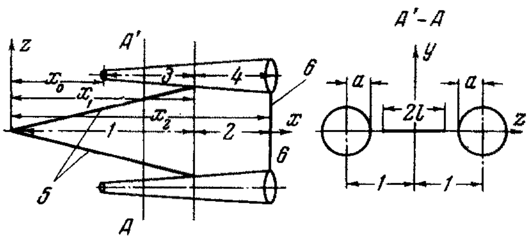

Fig. 1.

of the wing. The notation of dimensions will be found on Fig.1. The unit of length was selected as the half-distance between the axes of the nacelles, with respect to which all other dimensions are normalized (*a*, *l*, *x*, etc.)

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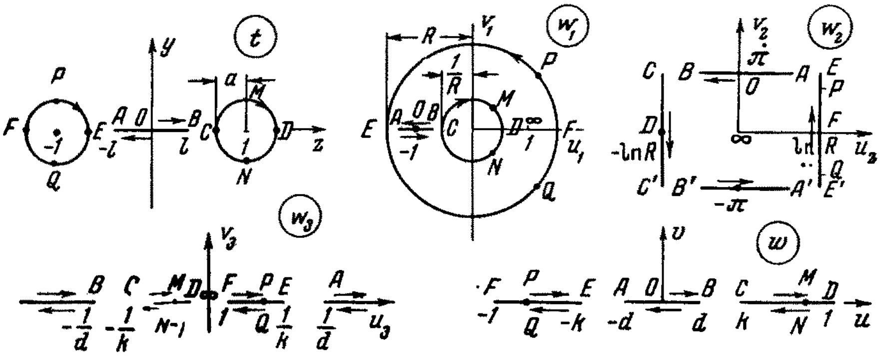

Fig. 2.

In general when $l \leqslant 1 - a$, the contour of the lateral cross-section of this system is transformed by conformal mapping on three segments of the real axis of the plane w:

$$-1 \leqslant w \leqslant -k, \quad -d \leqslant w \leqslant d, \quad k \leqslant w \leqslant 1, \quad d \leqslant k \tag{1}$$

by means of the function

$$w = -\left\{ \mathrm{sn} \left[\frac{1}{C} \ln \frac{t - \sqrt{1-a^2}}{t + \sqrt{1-a^2}} \right] \right\}^{-1} \tag{2}$$

where sn denotes the elliptic sine function.

Constants of transformation k, C and d are determined from the following equations

$$\frac{K(k)}{K(k')} = \frac{1}{\pi} \ln \frac{1 + \sqrt{1-a^2}}{a}, \quad k' = \sqrt{1-k^2}$$

$$C = \frac{\pi}{K(k')}, \quad d = w(t= \tag{3}$$

Here $K(k)$ and $K(k')$ represent complete elliptic integrals of the first kind with conjugate moduli k and k', a and l represent the dimensionless radius of the nacelles and the half-span of the wing in the cross-section x = const.

Fig.2 shows planes of intermediary transformations:

$$w_1 = \frac{t - \sqrt{1-a^2}}{t + \sqrt{1-a^2}}, \quad w_2 = \ln w_1, \quad w_3 = \mathrm{sn}\left(\frac{w_2}{C}, k\right), \quad w = -\frac{1}{w_3}$$

The complex velocity potential of the flow without circulation set up by the isolated wing (1) located normal to a stream having velocity A, is determined from Chaplygin's [2] formula.

$$\Phi(w) = -iA \int_w^1 \frac{\tau\left[\tau^2 - d^2 - (1-d^2)E(n)/K(n)\right]}{\sqrt{(\tau^2-1)(\tau^2-k^2)(\tau^2-d^2)}}\, d\tau, \qquad n = \sqrt{\frac{1-k^2}{1-d^2}} \tag{4}$$

The velocity potential due to the segment $-d \leqslant w \leqslant d$, which represents the wing, is

$$\varphi(w) = \operatorname{Re}\Phi = \pm A\left\{\sqrt{k^2 - w^2}\cos\psi + \sqrt{1 - d^2}\left[E(n, \psi) - \frac{E(n)}{K(n)}F(n, \psi)\right]\right\}$$

$$\psi = \arcsin\sqrt{\frac{1 - d^2}{1 - w^2}} \tag{5}$$

The corresponding velocity potential due to segments $-1 \leqslant w \leqslant -k$, $k \leqslant w \leqslant 1$ which represent the nacelles is

$$\varphi^\circ(w) = \varphi^\circ(-w) = \pm A\sqrt{1 - d^2}\left[E(n, \psi^\circ) - \frac{E(n)}{K(n)}F(n, \psi^\circ)\right] \tag{6}$$

$$\psi_0 = \arcsin\sqrt{\frac{1 - w^2}{1 - k^2}}$$

Here and below the plus sign in formulas (and also in indices) pertains to upper and minus sign to lower surfaces; $F(n, \psi)$ and $E(n, \psi)$ are elliptic integrals of the first and second kind, respectively; $F(n) = F(n, \frac{1}{2}\pi)$, $E(n) = E(n, \frac{1}{2}\pi)$. Velocities in planes t and w are related as follows

$$\frac{d\Phi}{dw} = \frac{d\Phi}{dt}\,\frac{dt}{dw}$$

For a wing without yaw $d\Phi / dt = -iV\alpha$ for $t = \infty$ and

$$A = \frac{2V\alpha}{C}\sqrt{1 - a^2}$$

where V and α are velocity and angle of attack of the undisturbed stream at a far distance.

For the calculation of aerodynamic forces acting on the system here considered we use the relation between the coefficient of pressure and velocity potential stemming from linear theory.

$$\frac{p_- - p_+}{{}^1/_2\,\rho V^2} = \frac{4}{V}\,\frac{\partial\varphi_+}{\partial x}$$

and we introduce special functions I for the wing and I^0 for the nacelles

$$I(a, l) = \frac{1}{2V\alpha}\int_0^l \varphi\,dz, \quad I^\circ(a, l) = \frac{1}{2V\alpha}\int_0^\pi \varphi^\circ \sin\theta\,d\theta \tag{7}$$

where θ is the polar angle in the plane $x =$ const. measured from the z-axis and the origin is on the axis of the nacelle.

Values of these functions are listed in Table 1 and on Figs. 3 and 4. Table 1 also contains values of transformation constants k and C.

TABLE 1

l	I	I^*	l	I	I^*	l	I	I^*
$a=0.11$			$a=0.21$			$a=0.32$		
$k=0.643$		$C=1.623$	$k=0.423$		$C=1.360$	$k=0.259$		$C=1.130$
0	0	0.175	0	0	0.331	0	0	0.509
0.100	0.004	0.175	0.100	0.004	0.333	0.100	0.004	0.513
0.200	0.016	0.177	0.200	0.017	0.339	0.200	0.019	0.523
0.300	0.035	0.181	0.300	0.038	0.349	0.300	0.043	0.542
0.400	0.063	0.190	0.400	0.069	0.364	0.400	0.079	0.570
0.500	0.100	0.201	0.500	0.108	0.386	0.500	0.130	0.618
0.600	0.148	0.219	0.600	0.164	0.418	0.600	0.206	0.698
0.700	0.203	0.247	0.700	0.232	0.493	0.650	0.264	0.773
0.800	0.273	0.302	0.750	0.286	0.569	0.680	0.564	1.180
0.850	0.321	0.357	0.775	0.330	0.623			
0.875	0.363	0.419	0.790	0.546	0.925	$a=0,\ k=1,\ C=2$		
0.890	0.503	0.650	—	—	—	—	$^1/_8\,\pi\, l^2$	0
$a=0.40$			$a=0.50$			$a=0.60$		
$k=0.174$		$C=0.995$	$k=0.093$		$C=0.840$	$k=0.044$		$C=0.695$
0	0	0.650	0	0	0.856	0	0	1.032
0.100	0.005	0.653	0.100	0.005	0.865	0.100	0.006	1.051
0.200	0.022	0.668	0.200	0.025	0.888	0.200	0.029	1.083
0.300	0.049	0.694	0.300	0.059	0.932	0.300	0.077	1.142
0.400	0.092	0.739	0.400	0.118	1.009	0.350	0.114	1.222
0.500	0.160	0.813	0.450	0.165	1.084	0.375	0.149	1.263
0.550	0.208	0.880	0.475	0.219	1.155	0.400	0.456	1.788
0.575	0.249	0.930	0.500	0.520	1.608	—	—	—
0.600	0.555	1.375	—	—	—	—	—	—
$a=0.70$			$a=0.80$			$a=0.90$		
$k=0.017$		$C=0.570$	$k=0.003$		$C=0.440$	$k=0$		$C=0.297$
0	0	1.260	0	0	1.520	0	0	1.845
0.100	0.009	1.278	0.100	0.012	1.545	0.050	0.005	1.863
0.200	0.040	1.335	0.150	0.032	1.593	0.075	0.012	1.886
0.250	0.072	1.398	0.175	0.050	1.640	0.100	0.147	2.390
0.275	0.099	1.450	0.200	0.272	2.185	$a=1,$	$k=0,$	$C=0$
0.300	0.376	1.995	—	—	—	—	0	2.580

Lift and pitching moment coefficients of the front part of the wing $C_y^{(1)}$ and $m_z^{(1)}$ and of the leading part of the nacelles $C_y^{(3)}$ and $m_z^{(3)}$ are determined from the formulas

$$C_y^{(1)} = \frac{4b^2}{S} 4\alpha \left[\frac{1}{8}\pi l_0^2 - I(a_0, l_0) + I(a_1, 1 - a_1)\right] \tag{8}$$

$$m_z^{(1)} = \frac{4b^3}{SL} 4\alpha \left\{x_1 I(a_1, 1 - a_1) - x_0\left[I(a_0, l_0) - \frac{1}{8}\pi l_0^2\right] - \int_0^{x_1} I(a, l)\,dx\right\} \tag{9}$$

$$C_y^{(3)} = \frac{4b^2}{S} 2\alpha \left[a_1 I^\circ(a_1, 1 - a_1) - a_0 I^\circ(a_0, l_0) - \int_{a_0}^{a_{\max}} I^\circ(a, l)\,da + \int_{a_1}^{a_{\max}} I^\circ(a, l)\,da\right] \tag{10}$$

$$m_z^{(3)} = \frac{4b^3}{SL} 2\alpha \left[x_1 a_1 I^\circ(a_1, 1 - a_1) - x_0 a_0 I^\circ(a_0, l_0) - \right.$$

$$\left. - \int_{x_0}^{x(a_{\max})} \left(a + x\frac{da}{dx}\right) I^\circ(a, l)\,dx - \int_{x(a_{\max})}^{x_1} \left(a + x\frac{da}{dx}\right) I^\circ(a, l)\,dx\right] \tag{11}$$

Here $2b$ is the distance between axes of the nacelles, L and S are the characteristic length and area equal both for the wing and for the nacelles, $4b^2/S$ is the aspect ratio of the combination of wing and nacelles, x_0 is the coordinate of the nacelle inlets, $a_0 = a(x_0)$, $l_0 = l(x_0)$, x_1 is the dimensionless length of the front part of the wing, $a_1 = a(x_1)$, $a_{\max}$ is the maximum relative radius of the front part of the nacelles.

In formulas (10) and (11) $l\,x)$ and (x) or $(l_1\,a)$] in the first integrals represent equations of the leading edge of the wing and of the contour of the nacelles up to the maximum section of the frontal part, while in second integrals they describe the same parameters behind that section.

In the case of sharp-nosed nacelles $a_0 = 0$, $I^0(0, l = 0$, $I(0, l_0) = 1/8\,\pi\, l_0^2$. In the case of nacelles with internal flow ($a_0 \neq 0$), forces acting from within the nacelle are not considered. When $a = 0$, formulas (8) and (9) reduce to known expressions for aerodynamic coefficients of a low aspect ratio wing as determined from slender body theory.

For the central part of the combination here considered we have $l = 1 - a$, and its aerodynamic coefficients are determined from the following formulas.

$$C_y^{(2)} + 2C_y^{(4)} = \frac{4b^2}{S} 4\alpha \left\{I(a_2, 1 - a_2) + a_2 I^\circ(a_2, 1 - a_2) - \right.$$

$$\left. - I(a_1, 1 - a_1) - a_1 I^\circ(a_1, 1 - a_1) + \int_{a_1}^{a_2} [G(a) - I^\circ(a, 1 - a)]\,da\right\} \tag{12}$$

$$m_z^{(2)} + 2m_z^{(4)} = \frac{4b^3}{SL} 4\alpha \Big\{ x_2 \left[I(a_2, 1 - a_2) + a_2 I^\circ (a_2, 1 - a_2) \right] -$$

$$- x_1 \left[I(a_1, 1 - a_1) + a_1 I^\circ (a_1, 1 - a_1) \right] - \int_{x_1}^{x_2} \Big[I(a, 1 - a) - xG(a) \frac{da}{dx} +$$

$$+ \left(a + x \frac{da}{dx} \right) I^\circ (a, 1 - a) \Big] dx \Big\} \qquad (13)$$

$$G(a) = \frac{1}{C} \sqrt{(1 - a^2)(1 - k^2)}$$

Here $C_y^{(2)}$ and $m_Z^{(2)}$ are aerodynamic coefficients of the central part of the wing, $C_y^{(4)}$ and $m_Z^{(4)}$ refer to the central part of the nacelles, x_2 = base coordinate of the nacelle $a_2 = a(x_2)$.

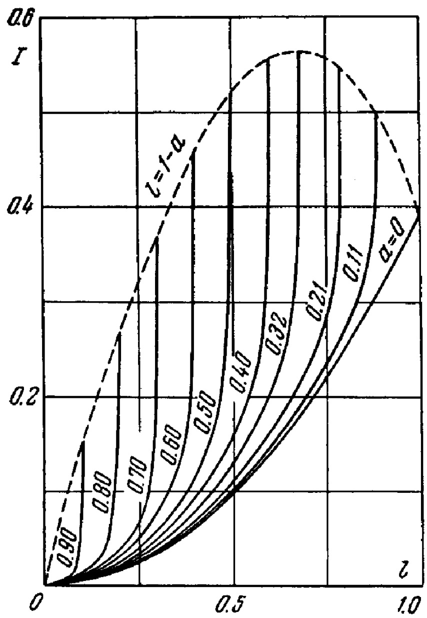

Fig. 3.

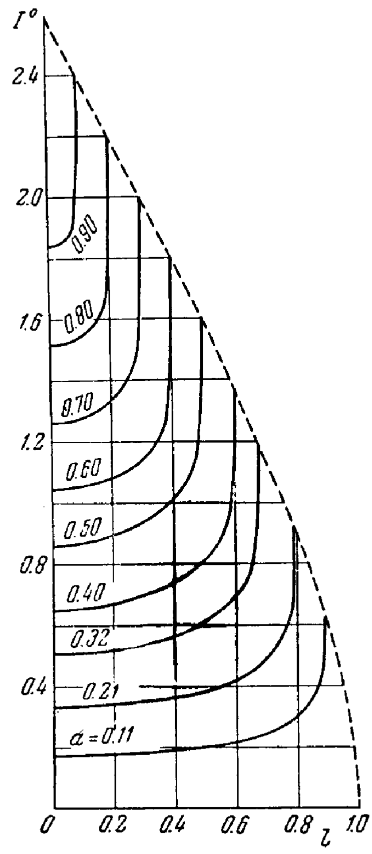

Fig. 4.

The aerodynamic coefficients of the entire combination are obtained by summation of the corresponding coefficients of its components. The analysis of obtained results indicates the following:

1. The presence of nacelles at the ends of the wing usually increases the lifting force of the combination by comparison with an isolated wing having equal aspect ratio $\lambda = 4b^2/S$. (Here S = surface plan area of an isolated wing, b = half wingspan).

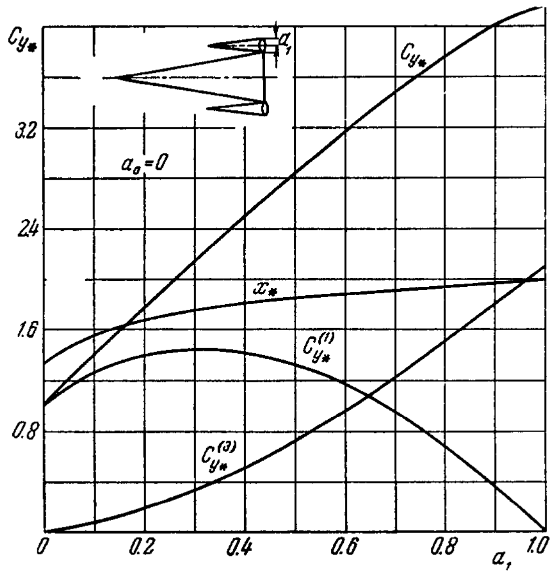

Fig. 5.

As an example Fig.5 shows the aerodynamic characteristics of the front part of a combination consisting of wing and pointed nacelles at its ends ($a_0 = 0$, $da/dx \geqslant 0$); C_{*y} is the ratio of coefficients of the lifting force of wing with and without nacelles, $C_{y*}^{(1)}$, $C_{*y}^{(3)}$ are relative coefficients of the lifting force of separate parts of the combination (of the wing in presence of nacelles and of nacelles in presence of wing); x_* is coordinate of the center of pressure of the leading part of triangular wing with nacelles at the ends, referred to its length x_1

$$C_{y_*} = \frac{C_y^{(1)} + 2C_y^{(3)}}{{}^1/_2\,\pi\lambda\alpha}, \qquad C_{y_*}^{(1)} = \frac{C_y^{(1)}}{{}^1/_2\,\pi\lambda\alpha}$$

$$C_{y_*}^{(3)} = \frac{C_y^{(3)}}{{}^1/_2\,\pi\lambda\alpha}, \qquad x_* = \frac{m_z^{(1)} + 2m_z^{(3)}}{C_y^{(1)} + 2C_y^{(3)}}\,\frac{L}{x_1}$$

In the case of pointed divergent nacelles the aerodynamic characteristics of the leading part of the combination, C_y and M_z, are determined in terms of the dimensionless a_1 of the nacelles at the point of contact with the wing. The effect of the shape of wing and of the frontal part on the nacelles on the magnitude of the lift coefficient does not exceed 5 per cent. In the case of nacelles with internal flow ($a_0 \neq 0$), their shape exerts a major effect on the lifting force of the combination; it decreases with increasing ratio a_0/a_1. However, in the case of cylindrical nacelles also when $a \leqslant 0.6$, one encounters a considerable increase of the lifting force of the combination by comparison with an isolated wing.

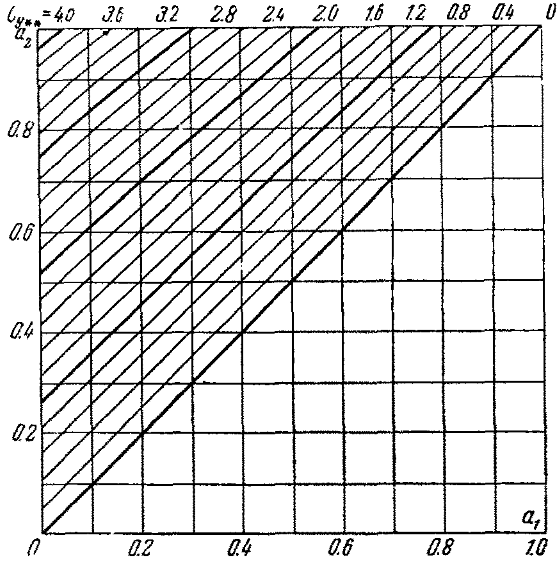

Fig. 6.

The location of the center of pressure in the case of nacelles with internal flow depends only slightly on their shape and is determined mainly by the value of a_1.

2. The lifting force of the central part of the wing with nacelles does not depend on their shape but is determined solely by the magnitude of their relative radii a_1 in the front and a_2 in the rear. The lifting force of the central part is positive when $a_2 > a_1$, and negative when $a_2 < a_1$, and equals zero when $a_2 = a_1$. Fig.6 presents the ratio of lift coefficients of the central part of the combination and of an isolated wing of aspect ratio $\lambda = 4b^2/S$:

$$C_{y_{**}} = \frac{C_y^{(2)} + 2C_y^{(4)}}{{}^1/_2\, \pi\lambda\alpha}$$

$$(C_{y_{**}}(a_1, a_2) = -C_{y_{**}}(a_2, a_1))$$

The shape of the central part shows a direct influence on the location of the center of pressure. Its widening $a_2 > a_1$ and increase of length displaces the center of pressure toward rear edge; its narrowing $a_2 < a_1$ and decrease of length displaces the center of pressure forward.

3. The distribution of loading on the wing and on nacelles dC_y/dx

indicates that their mutual interaction increases with approach to the central part, in the vicinity of which the load on the wing increases rapidly by comparison with that on an isolated wing of a similar shape.

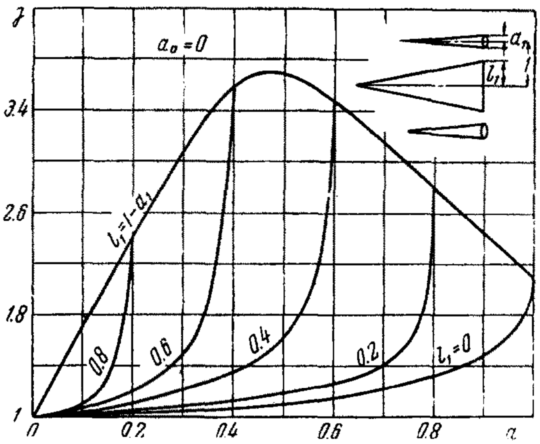

Fig. 7.

4. Fig. 7 shows the ratio of the lifting force of the combination of the wing with pointed divergent nacelles, in the general case of the presence of a separation between their bases and the wing ($l_1 \to 1 - a_1$), to the sum of lifting forces of these nacelles and the wing performing separately from each other:

$$J = \frac{[C_y^{(1)} + 2C_y^{(3)}]\, S}{2\pi\alpha\,(l_1^2 + 2a_1^2)}$$

It follows from Fig. 7 that the interaction of the wing with nacelles leads to the enhancement of lifting properties of their surfaces by comparison with isolated elements.

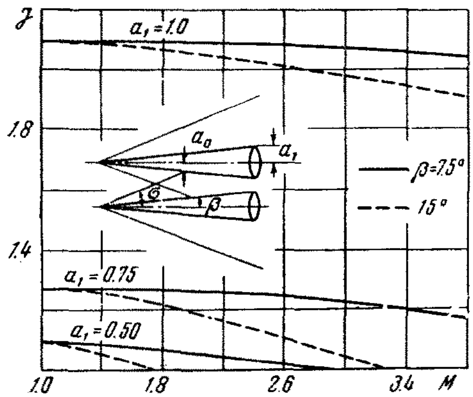

Fig. 8.

When the distance between them tends to zero: $l_1 \longrightarrow 1 - a_1$, the interaction increases rapidly. The case $l = 0$ corresponds to two bodies of revolution without a wing, for which the lifting force is independent of their shape and is determined merely by the relative radius a_1 of their bases.

5. In the case of supersonic velocities of flight, slender body theory is applicable to the part of the combination which is located within the region of disturbance spreading from its apex. If various elements of the combination are located outside the region disturbed by neighboring bodies, aerodynamic forces acting on them are the same as in case of isolated bodies. Considering this circumstance the ratio of lifting forces of two mutually interacting pointed bodies of revolution, whose axes are parallel and whose noses are in a plane normal to the axes, to the lifting force of two isolated bodies equals

$$J = \left(\frac{a_0}{a_1}\right)^2 + \frac{4}{\pi a_1^2}\left\{a_1 I^\circ(a_1, O) - a_0 I^\circ(a_0, Q) - \int_{a_0}^{a_1} I^\circ(a, O)\, da\right\}$$

where a_0 is the relative radius of the bodies in that cross-section where their mutual interaction begins. Fig.8 shows this ratio for two cones having vertex angles β, as a function of Mach number M of the oncoming stream. The angle σ of the shock at the vertex of the cone, with its axis, is assumed to be the same as in the case of an axisymmetric flow; the distortion of the shock in the region disturbed by the neighboring cone is neglected. The cones do not interact when

$$\sigma \leqslant \operatorname{arc\,tg}\left(\frac{2}{a_1} - 1\right)\operatorname{tg}\beta$$

When the bases of cones touch each other $a_1 = 1.0$, and the cones interact for an arbitrary value of M.

In analogous manner one can determine the aerodynamic characteristics of the combination of wing and necelles, separate elements of which do not interact.

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